Int. J. Solids Structures, 1974, 10, pp. 77–92. Pergamon Press. Printed in Gt. Britain.

A DISPLACEMENT BOUNDING PRINCIPLE IN SHAKEDOWN OF STRUCTURES SUBJECTED TO CYCLIC LOADS

MICHELE CAPURSO

Facoltà d'Ingegneria, Università di Bologna, Italy

(*Received* 24 *January* 1973; *revised* 12 *June* 1973)

Abstract—Continua or structures composed of elastic perfectly-plastic material subjected to cyclic loads which vary within the shakedown limits are considered. A theorem bounding the residual deflection at any point is presented.

Some numerical examples are discussed.

1. INTRODUCTION

The problem of structures subjected to several static loading conditions, each of which can vary independently between certain limits in the elastic–plastic range, was first studied by Melan[1].

In Melan's paper it was pointed out that the overall deformation of the structure may increase indefinitely in the loading program, even though no single load condition causes plastic collapse. This characteristic feature of failure is avoided if, and only if, plastic strains do not recur after the earlier stages of the loading program. Hence the structure, after some finite amount of plastic strains, can respond, in a purely elastic manner, following a residual stress distribution correspondent to the above mentioned permanent strains.

The structure is then said to shakedown and the maximum loading under which the structure can shakedown is called the "shakedown loading."

In recent years, a great deal of attention has been focused on the estimation of the shakedown loading for elastic–plastic structures.

A state-of-the-art on this subject as of 1960 is given in the excellent survey by Koiter[2] on the theory of elastic-plastic solids.

Considerable progress in the practical application of the shakedown theory has been made in recent years with the help of the concepts of mathematical programming. In this context the linear programming theory has provided both efficient tools for numerical solutions and a new mathematical framework for some features of the shakedown theory (see e.g. [3, 4]).

At this point, the only major limitation to the practical application of the present theory is the absence of general bounding procedures which enable us to evaluate the local plastic deformations when the shakedown loading is approached.

In fact, when the loading conditions of the structure vary within the limits for which the structure shakes-down but outside the initial elastic loading limits, the permanent damage caused by the initial cycles of plastic strains may be intolerable.

Then a realistic assessment of the structural safety requires an *a priori* determination of the order of magnitude of local quantities, such as displacements or plastic strains. Unfortunately, these values depend on the loading history which is, as a rule, unknown. Then we are obliged to resort to bounding procedures.

Some of these bounding techniques have been developed recently[5, 6] for discrete structural models with piecewise linear yield surfaces.

In this context, upper bounds on the requested local quantities may be evaluated by solving mathematical programming problems which may be considered as general bounding procedures in the field of discrete models of continua or structures.

More recently Ponter[7] has derived a general principle which allows the evaluation of upper bounds to local displacement of elastic–plastic structures subjected to variable loading.†

In this work the bound is expressed as the sum of the deflections that would have occurred if the structure was purely elastic and additional displacement that may be derived from the elastic strain energy of a residual state of stress.

In this paper an entirely different derivation of a new principle bounding from above the residual local displacement of an elastic–plastic structure subject to variable loading, is proposed.

The result is obtained by using two additional statements (see sections 3 and 4) restricting the complementary and the direct plastic work of the structure performed during any loading history from the undisturbed state up to the time τ at which the work-bounds are desired.

By using these results for a particular loading programme which includes both the actual loading path up to $t = \tau$, and a subsequent "fictitious loading path" from $t = \tau$ up to $t = T$, the general bounding principle (see section 5) is obtained.

The bound is so expressed as the sum of two displacements which may be both derived from the elastic strain energy of two residual states of stress; the first one of these fields is associated with the "fictitious loading condition" at the time $t = T$, the second one to the actual loading path performed from the undisturbed state at $t = 0$ up to the time $t = \tau$ at which the displacement bound is required.

In section 6, two simple examples are discussed; as in the aforementioned work of Ponter, it is seen that the bounds are generally not very accurate but provide a simple calculation which may constitute a useful additional information in assessing the structural safety.

Possible links with other previous and recent works on related topics are pointed out in the text and in the concluding remarks.

2. FUNDAMENTALS

Let us consider a continuum of volume V and surface area S subjected, at any instant t, to surface tractions T_i on the unconstrained region S_T of S and to body forces X_i throughout its volume V.

Let us suppose that the material behavior is elastic–perfectly plastic of the standard type (i.e. Drucker's stability postulate[8] holds) and that the elastic stress–strain relationships are linear (i.e. Hooke's law holds).

With these assumptions, the actual displacement u_i at any instant t may be considered

† The work of Ponter was brought to the author's attention after the completion of the first version of this paper.

as the sum of the displacement u_i^E in a purely linear elastic regime under the same load condition, and of the path-dependent residual displacement u_i^R caused by the plastic-strains that the structure has previously suffered.

Thus we have:

$$u_i(t) = u_i^E(t) + u_i^R(t). \tag{1}$$

In the same spirit, the actual stresses $\sigma_{ij}(t)$ and the actual total strains $\varepsilon_{ij}(t)$ may be conceived as split into the following addends:

$$\begin{aligned} \sigma_{ij}(t) &= \sigma_{ij}^E(t) + \sigma_{ij}^R(t) \\ \varepsilon_{ij}(t) &= \varepsilon_{ij}^E(t) + \varepsilon_{ij}^R(t) + \varepsilon_{ij}^P(t) \end{aligned} \tag{2}$$

where σ_{ij}^E and ε_{ij}^E denote the purely-elastic stress and strain response of the structure under the same load conditions, and σ_{ij}^R, ε_{ij}^R denote the elastic response of the structure to the non-compatible plastic strain distribution ε_{ij}^P.

Then it is obvious that at any instant t the following conditions must be satisfied:

(a) *Equilibrium equations*

$$\begin{aligned} &\sigma_{ij/i}^E + X_j = 0, \qquad \sigma_{ij/i}^R = 0 \quad \text{in} \quad V \\ &\sigma_{ij}^F n_i = T_j, \qquad \sigma_{ij}^R n_i = 0 \quad \text{in} \quad S_T. \end{aligned} \tag{3}$$

(b) *Compatibility equations*

$$\begin{aligned} &\varepsilon_{ij}^E = \tfrac{1}{2}(u_{i/j}^E + u_{j/i}^E), \qquad \varepsilon_{ij}^R + \varepsilon_{ij}^P = \tfrac{1}{2}(u_{i/j}^R + u_{j/i}^R) \quad \text{in} \quad V \\ &u_i^E = 0, \qquad u_i^R = 0 \quad \text{in} \quad S - S_T. \end{aligned} \tag{4}$$

(c) *Elasticity equations*

$$\varepsilon_{ij}^E = A_{ijhk}\sigma_{hk}^E, \qquad \varepsilon_{ij}^R = A_{ijhk}\sigma_{hk}^R \tag{5}$$

where the tensor of elastic coefficients A_{ijhk} has the usual properties of symmetry:

$$A_{ijhk} = A_{jihk} = A_{ijkh} = A_{hkij}. \tag{6}$$

As known, Melan's theorem states that if any time-independent distribution of residual stresses $\bar{\sigma}_{ij}^R$ can be found such that the state of stress:

$$\bar{\sigma}_{ij}(t) = \sigma_{ij}^E(t) + \bar{\sigma}_{ij}^R \tag{7}$$

is at any time t a safe state, the structure, after an initial elastic–plastic behavior, shakes-down and the response to subsequent load variations is purely elastic.

On the assumption that the structure shakes-down, we can now proceed to prove some general theorems which will enable us to develop subsequently the desired displacement bounding principle.

3. A COMPLEMENTARY PLASTIC-WORK THEOREM

Let us denote by $U_P(\tau)$ the quantity:

$$U_P(\tau) = \int_{S_T} \mathrm{d}S \int_0^\tau \dot{T}_i(t) u_i^R(t)\,\mathrm{d}t + \int_V \mathrm{d}V \int_0^\tau \dot{X}_i(t) u_i^R(t)\,\mathrm{d}t \tag{8}$$

which may be referred to as "total complementary plastic work" performed during a load path starting from the undisturbed state at $t = 0$ up to $t = \tau$.

By virtue of equations (2–4), the right hand side of (8) can be rewritten in the following form:

$$\begin{aligned} U_P(\tau) &= \int_V \mathrm{d}V \int_0^\tau \dot{\sigma}_{ij}(t)\varepsilon_{ij}^R(t)\,\mathrm{d}t + \int_V \mathrm{d}V \int_0^\tau \dot{\sigma}_{ij}(t)\varepsilon_{ij}^P(t)\,\mathrm{d}t \\ &= \int_V \mathrm{d}V \int_0^\tau \dot{\sigma}_{ij}^R(t)\varepsilon_{ij}^R(t)\,\mathrm{d}t + \int_V \mathrm{d}V \int_0^\tau \dot{\sigma}_{ij}(t)\varepsilon_{ij}^P(t)\,\mathrm{d}t \end{aligned} \tag{9}$$

after using the condition:

$$\int_V \dot{\sigma}_{ij}^E(t)\varepsilon_{ij}^R(t)\,\mathrm{d}V = \int_V \sigma_{ij}^R(t)\dot{\varepsilon}_{ij}^E(t)\,\mathrm{d}V = 0. \tag{10}$$

With the aid of equations (5), we may write equation (9) in the form:

$$U_P(\tau) = \frac{1}{2}\int_V A_{ijhk}\sigma_{ij}^R(\tau)\sigma_{hk}^R(\tau)\,\mathrm{d}V + \int_V \sigma_{ij}(\tau)\varepsilon_{ij}^P(\tau)\,\mathrm{d}V - \int_V \mathrm{d}V \int_0^\tau \sigma_{ij}(t)\dot{\varepsilon}_{ij}^P(t)\,\mathrm{d}t \tag{11}$$

where $\sigma_{ij}(\tau)$, $\sigma_{ij}^R(\tau)$, $\varepsilon_{ij}^P(\tau)$ represent the stress components, the self-stresses and the plastic strains at the final stage of the assigned load path.

We can now demonstrate the following bounding principle:

Theorem I

If σ_{ij}^{R*} *represents any distribution of self-stresses such that the state of stress*

$$\sigma_{ij}^*(\tau) = \sigma_{ij}^E(\tau) + \sigma_{ij}^{R*} \tag{12}$$

is a safe state, the complementary plastic work $U_P(\tau)$ *may be upperbounded by the condition:*

$$U_P(\tau) \le \frac{1}{2}\int_V A_{ijhk}\sigma_{ij}^{R*}\sigma_{hk}^{R*}\,\mathrm{d}V. \tag{13}$$

To prove this principle, let us express any admissible distribution of self-stresses as:

$$\sigma_{ij}^{R*} = \sigma_{ij}^R(\tau) + \Delta\sigma_{ij}^R \tag{14}$$

and indicate by $R(\tau)$ the difference:

$$R(\tau) = \frac{1}{2}\int_V A_{ijhk}\sigma_{ij}^{R*}\sigma_{hk}^{R*}\,\mathrm{d}V - U_P(\tau). \tag{15}$$

From equations (11) and (15) we then may write:

$$\begin{aligned} R(\tau) = &\int_V A_{ijhk}\,\Delta\sigma_{ij}^R\sigma_{hk}^R(\tau)\,\mathrm{d}V + \frac{1}{2}\int_V A_{ijhk}\,\Delta\sigma_{ij}^R\,\Delta\sigma_{hk}^R\,\mathrm{d}V \\ &- \int_V \sigma_{ij}(\tau)\varepsilon_{ij}^P(\tau)\,\mathrm{d}V + \int_V \mathrm{d}V \int_0^\tau \sigma_{ij}(t)\dot{\varepsilon}_{ij}^P(t)\,\mathrm{d}t. \end{aligned} \tag{16}$$

On the other hand, the first integral on the right hand side of equation (16) may be written in the form:

$$\int_V \Delta\sigma_{ij}{}^R \varepsilon_{ij}{}^R(\tau)\,\mathrm{d}V = -\int_V \Delta\sigma_{ij}{}^R \varepsilon_{ij}{}^P(\tau)\,\mathrm{d}V \tag{17}$$

which, making use of the condition:

$$\Delta\sigma_{ij}{}^R = \sigma_{ij}{}^*(\tau) - \sigma_{ij}(\tau) \tag{18}$$

reduces to:

$$-\int_V \{\sigma_{ij}{}^*(\tau) - \sigma_{ij}(\tau)\}\varepsilon_{ij}{}^P(\tau)\,\mathrm{d}V = \tag{19}$$

$$= -\int_V \mathrm{d}V \int_0^\tau \sigma_{ij}{}^*(\tau)\dot{\varepsilon}_{ij}{}^P(t)\,\mathrm{d}t + \int_V \sigma_{ij}(\tau)\varepsilon_{ij}{}^P(\tau)\,\mathrm{d}V. \tag{20}$$

Thus equation (16) may be finally rearranged in the form:

$$R(\tau) = \frac{1}{2}\int_V A_{ijhk}\,\Delta\sigma_{ij}{}^R\,\Delta\sigma_{hk}{}^R\,\mathrm{d}V + \int_V \mathrm{d}V \int_0^\tau \{\sigma_{ij}(t) - \sigma_{ij}{}^*(\tau)\}\dot{\varepsilon}_{ij}{}^P(t)\,\mathrm{d}t. \tag{21}$$

Since $\sigma_{ij}{}^*(\tau)$ is, by hypothesis, a safe state of stresses, Drucker's stability postulate assures that the second integral in equation (21) is always non-negative.

Then it is obvious that:

$$R(\tau) \geq 0 \tag{22}$$

and thus theorem I is proved.

It is noteworthy to observe that, by assuming a piecewise linear yield surface, theorem I may be directly deduced from Maier's theorems of bounding complementary plastic work[9], simply by cancelling the term depending on the hardening of material (see e.g. inequality (14) at page 264 of the above quoted work). However, in the particular context of perfect plasticity, the present conclusion is more general since it does not require any particular assumption on the yield surface except for that imposed by the Drucker stability postulate.

In this sense it may be interesting to note that the search via theorem I for the best upper bound $U_P{}^0$ obviously reduces to the minimization of the right hand side of (13) with $\sigma_{ij}{}^{R*}$ constrained by the condition that the state of stress (12) is contained within the yield surface.

This convex optimization problem bears a close resemblance to the Haar–Karman principle[10], and differs from the latter only in that we have used self-equilibrated states of stress instead of the global state of stress.

Then theorem I may be considered equivalent to another bounding principle derived by Hodge[11], and it may be inserted, as a particular case, in the context of a recent work by Ponter and Martin[12] on the connections between the flow and deformation theories of plasticity.

4. A DIRECT PLASTIC-WORK THEOREM

Let us denote by $W_P(\tau)$ the quantity:

$$W_P(\tau) = \int_{S_T} \mathrm{d}S \int_0^\tau T_i(t)\dot{u}_i{}^R(t)\,\mathrm{d}t + \int_V \mathrm{d}V \int_0^\tau X_i(t)\dot{u}_i{}^R(t)\,\mathrm{d}t \tag{23}$$

which may be referred to as "total direct plastic work" performed during the load path starting from the undisturbed state at $t = 0$ up to $t = \tau$.

By virtue of equations (2–4), the right hand side of (23) can be rewritten in the following form:

$$W_P(\tau) = \int_V \mathrm{d}V \int_0^\tau \sigma_{ij}(t)\{\dot{\varepsilon}_{ij}{}^R(t) + \dot{\varepsilon}_{ij}{}^P(t)\}\,\mathrm{d}t$$
$$= \int_V \mathrm{d}V \int_0^\tau \sigma_{ij}{}^E(t)\{\dot{\varepsilon}_{ij}{}^R(t) + \dot{\varepsilon}_{ij}{}^P(t)\}\,\mathrm{d}t = \int_V \mathrm{d}V \int_0^\tau \sigma_{ij}{}^E(t)\dot{\varepsilon}_{ij}{}^P(t)\,\mathrm{d}t \quad (24)$$

after using the relations:

$$\int_V \sigma_{ij}{}^R(t)\{\dot{\varepsilon}_{ij}{}^R(t) + \dot{\varepsilon}_{ij}{}^P(t)\}\,\mathrm{d}V = 0$$
$$\int_V \sigma_{ij}{}^E(t)\dot{\varepsilon}_{ij}{}^R(t)\,\mathrm{d}V = \int_V \dot{\sigma}_{ij}{}^R(t)\varepsilon_{ij}{}^E(t)\,\mathrm{d}V = 0. \quad (25)$$

We can now develop the following bounding principle:

Theorem II

If any time-independent distribution of residual stresses $\bar{\sigma}_{ij}{}^R$ *can be found such that the state of stresses:*

$$\bar{\sigma}_{ij}(t) = m\sigma_{ij}{}^E(t) + \bar{\sigma}_{ij}{}^R \quad (26)$$

associated with a factor $m > 1$, *is a safe state for any time* t *falling in the interval:*

$$0 \le t \le \tau, \quad (27)$$

then the direct plastic work $W_P(\tau)$ *may be upperbounded by the condition:*

$$W_P(\tau) \le \frac{1}{m-1} \cdot \frac{1}{2}\int_V A_{ijhk}\bar{\sigma}_{ij}{}^R\bar{\sigma}_{hk}{}^R\,\mathrm{d}V. \quad (28)$$

To prove this principle we can observe that from Drucker's stability postulate we have the inequality:

$$\{\sigma_{ij}(t) - \bar{\sigma}_{ij}(t)\}\dot{\varepsilon}_{ij}{}^P(t) \ge 0 \quad (29)$$

which, by means of equations (2) and (26), may be rewritten in the form:

$$(1-m)\sigma_{ij}{}^E(t)\dot{\varepsilon}_{ij}{}^P(t) + \{\sigma_{ij}{}^R(t) - \bar{\sigma}_{ij}{}^R\}\dot{\varepsilon}_{ij}{}^P(t) \ge 0. \quad (30)$$

Integrating (30) over the volume of the body and using the results of equation (24), we get the inequality:

$$(1-m)\dot{W}_P(t) + \int_V \{\sigma_{ij}{}^R(t) - \bar{\sigma}_{ij}{}^R\}\dot{\varepsilon}_{ij}{}^P(t)\,\mathrm{d}V \ge 0 \quad (31)$$

which, by means of the obvious condition:

$$\int_V \{\sigma_{ij}{}^R(t) - \bar{\sigma}_{ij}{}^R\}\dot{\varepsilon}_{ij}{}^P(t)\,\mathrm{d}V = -\int_V \{\sigma_{ij}{}^R(t) - \bar{\sigma}_{ij}{}^R\}_i\dot{\varepsilon}_j{}^R(t)\,\mathrm{d}V \quad (32)$$

may be rewritten in the form:

$$\dot{W}_P(t) \leq -\frac{1}{m-1}\int_V \{\sigma_{ij}{}^R(t) - \bar{\sigma}_{ij}{}^R\}\dot{\varepsilon}_{ij}{}^R(t)\,\mathrm{d}V. \tag{33}$$

Integration of this inequality with respect to time, from $t = 0$ to $t = \tau$, results in:

$$W_P(\tau) \leq \frac{1}{m-1}\left\{\frac{1}{2}\int_V A_{ijhk}\bar{\sigma}_{ij}{}^R\bar{\sigma}_{hkj}\,\mathrm{d}V - \frac{1}{2}\int_V A_{ijhk}[\sigma_{ij}{}^R(\tau) - \bar{\sigma}_{ij}{}^R][\sigma_{hk}{}^R(\tau) - \bar{\sigma}_{hk}{}^R]\,\mathrm{d}V\right\}. \tag{34}$$

The bounding condition (28) follows immediately from (34). Thus the theorem is proven.

At this point, it is interesting to note that the existence of a safety factor $S > 1$ with regard to shakedown is adequate to assure the existence of a bound for $W_P(\tau)$ at any time τ.

Then the search via theorem II for the best upper bound $W_P{}^0$ reduces to the minimization of the right hand side of (28) with $\bar{\sigma}_{ij}{}^R$ constrained by the condition that for any t, the state of stress (26) is contained within the yield surface, and m constrained by the obvious conditions:

$$1 \leq m \leq S. \tag{35}$$

It is easy to see that the existence of a factor $m > 1$ such that the state of stress $m\sigma_{ij}{}^E(t)$ is contained within the yield surface, is adequate to assure that the best bound $W_P{}^0$ is zero.

Then if $\alpha > 1$ is the factor for which at any time t the loading condition remains within the elastic limits, the interval (35) may be substituted by the inequality:

$$\alpha \leq m \leq S. \tag{36}$$

As a final point, it may be noted that the bounding condition (28) bears a close resemblance to Koiter's inequality (see e.g. [2], p. 208) ensuring that the overall plastic deformation is bounded when the structure shakes-down.

However, it is easy to see that the total plastic work bounded by Koiter is not coincident with the plastic work $W_P(\tau)$ which we have previously defined, and also that the constraints of the two bounding conditions are quite different.

5. THE DISPLACEMENT BOUNDING PRINCIPLE

Let us suppose that the body is subjected to the following load programme: first the surface tractions and the body forces change from $t = 0$ to $t = \tau$ along the actual load path; second, the surface tractions and the body forces change from $t = \tau$ to $t = T$ along the linear fictitious path:

$$\begin{aligned} T_i &= \frac{T_i^S - T_i(\tau)}{T-\tau}(t-\tau) + T_i(\tau) \qquad \tau \leq t \leq T \\ X_i &= \frac{X_i^S - X_i(\tau)}{T-\tau}(t-\tau) + X_i(\tau) \end{aligned} \tag{37}$$

where T_i^S and X_i^S represent any safe state of loading for the body.

Let us now evaluate the total complementary plastic work performed from the undisturbed state at $t = 0$ up to $t = T$. Obviously we have:

$$U_P(T) = U_P(\tau) + \int_{S_T} \mathrm{d}S \int_\tau^T \dot{T}_i(t) u_i^R(t)\,\mathrm{d}t + \int_V \mathrm{d}V \int_\tau^T \dot{X}_i(t) u_i^R(t)\,\mathrm{d}t$$

$$= U_P(\tau) + \int_{S_T} \mathrm{d}S \int_\tau^T \dot{T}_i(t)\{u_i^R(t) - u_i^R(\tau)\}\,\mathrm{d}t + \int_{S_T} \{T_i^S - T_i(\tau)\} u_i^R(\tau)\,\mathrm{d}S$$

$$+ \int_V \mathrm{d}V \int_\tau^T \dot{X}_i(t)\{u_i^R(t) - u_i^R(\tau)\}\,\mathrm{d}t + \int_V \{X_i^S - X_i(\tau)\} u_i^R(\tau)\,\mathrm{d}V. \qquad (38)$$

On the other hand, since we have:

$$U_P(\tau) + W_P(\tau) = \int_{S_T} T_i(\tau) u_i^R(\tau)\,\mathrm{d}S + \int_V X_i(\tau) u_i^R(\tau)\,\mathrm{d}V \qquad (39)$$

we may write equation (38) in the form:

$$U_P(T) = -W_P(\tau) + \int_S T_i^S u_i^R(\tau)\,\mathrm{d}S + \int_V X_i^S u_i^R(\tau)\,\mathrm{d}V + \Delta \qquad (40)$$

where:

$$\Delta = \int_{S_T} \mathrm{d}S \int_\tau^T \dot{T}_i(t)\{u_i^R(t) - u_i^R(\tau)\}\,\mathrm{d}t + \int_V \mathrm{d}V \int_\tau^T \dot{X}_i(t)\{u_i^R(t) - u_i^R(\tau)\}\,\mathrm{d}t. \qquad (41)$$

Since:

$$u_i^R(t) - u_i^R(\tau) = \int_\tau^t \dot{u}_i^R(t)\,\mathrm{d}t \qquad (42)$$

equation (41), being $\dot{T}_i(t)$ and $\dot{X}_i(t)$ constant in the interval $\tau \le t \le T$, may be rewritten as:

$$\Delta = \int_{S_T} \mathrm{d}S \int_\tau^T \mathrm{d}t \int_\tau^t \dot{T}_i(\xi) \dot{u}_i^R(\xi)\,\mathrm{d}\xi + \int_V \mathrm{d}V \int_\tau^T \mathrm{d}t \int_\tau^t \dot{X}_i(\xi) \dot{u}_i(\xi)\,\mathrm{d}\xi. \qquad (43)$$

On the other hand, we may write:

$$\int_{S_t} \dot{T}_i(\xi) \dot{u}_i^R(\xi)\,\mathrm{d}\xi + \int_V \dot{X}_i(\xi) \dot{u}_i^R(\xi)\,\mathrm{d}\xi = \int_V \dot{\sigma}_{ij}(\xi) \dot{\varepsilon}_{ij}^R(\xi)\,\mathrm{d}V + \int_V \dot{\sigma}_{ij}(\xi) \dot{\varepsilon}_{ij}^P(\xi)\,\mathrm{d}V$$

$$= \int_V \dot{\sigma}_{ij}^R(\xi) \dot{\varepsilon}_{ij}^R(\xi)\,\mathrm{d}V = \int_V A_{ijhk} \dot{\sigma}_{ij}^R(\xi) \dot{\sigma}_{hk}^R(\xi)\,\mathrm{d}V \ge 0 \qquad (44)$$

Thus, equation (43) yields the inequality:

$$\Delta \ge 0 \qquad (45)$$

which, once inserted in equation (40), enables us to affirm that:

$$\int_{S_T} T_i^S u_i^R(\tau)\,\mathrm{d}S + \int_V X_i^S u_i^R(\tau)\,\mathrm{d}V \le U_P(T) + W_P(\tau). \qquad (46)$$

Then, making use of the theorems previously developed in sections 3 and 4, we can formulate the following bounding principle:

Theorem III

The residual displacements occurring in any loading programme starting from the undisturbed state at $t = 0$ up to any time $t = \tau$, may be bounded by the inequality:

$$\int_{S_T} T_i^S u_i^R(\tau)\,\mathrm{d}S + \int_V X_i^S u_i^R(\tau)\,\mathrm{d}V \leq \frac{1}{2}\int_V A_{ijhk}\sigma_{ij}^{R*}\sigma_{hk}^{R*}\,\mathrm{d}V + \frac{1}{m-1}\cdot\frac{1}{2}\int_V A_{ijhk}\bar{\sigma}_{ij}^{R}\bar{\sigma}_{hk}^{R}\,\mathrm{d}V \tag{47}$$

where σ_{ij}^{R*} represents any distribution of self-stresses which enable us to compress inside the yield limits the elastic stress state associated with the fictitious loads T_i^S and X_i^S, and $\bar{\sigma}_{ij}^R$ represents any distribution of self-stresses which renders the state of stress associated with a factor $m > 1$:

$$m\sigma_{ij}^E(t) + \bar{\sigma}_{ij}^R$$

a safe state for any t falling in the interval:

$$0 \leq t \leq \tau.$$

It may be noted that the choice of T_i^S, X_i^S is dictated entirely by those properties of the displacement field $u_i^R(\tau)$ for which a bound is desired. Thus we can make deliberate use of the fact that T_i^S, X_i^S and $u_i^R(\tau)$ may be completely independent of each other.

Therefore, if we suppose that the body forces X_i^S vanish and that T_i^S is a single point load R^S acting at a point P along a prefixed direction a, equation (47) reduces to:

$$u_a^R(P) \leq \frac{1}{2R^S}\left\{\int_V A_{ijhk}\sigma_{ij}^{R*}\sigma_{hk}^{R*}\,\mathrm{d}V + \frac{1}{m-1}\int_V A_{ijhk}\bar{\sigma}_{ij}^{R}\bar{\sigma}_{hk}^{R}\,\mathrm{d}V\right\} \tag{48}$$

where $u_a^R(P)$ is the residual displacement component of the point P in the direction a. Then the search via (48) for the best upper bound $u_a^{R0}(P)$ reduces to the minimization of the right hand side of (48) under the appropriate constraints for R^S, σ_{ij}^{R*}, $\bar{\sigma}_{ij}^R$ and m.

This optimization problem may be solved first by substituting the last term of (48) with the best plastic work bound W_P^0, and then by solving the minimum problem:

$$\text{minimize:}\quad \frac{1}{2R^S}\left\{\int_V A_{ijhk}\sigma_{ij}^{R*}\sigma_{hk}^{R*}\,\mathrm{d}V + 2W_P^0\right\} \tag{49}$$

under the constraints that R^S be less than the static collapse load R^C, and that the stress state $\sigma_{ij}^E + \sigma_{ij}^{R*}$ (sum of the fully elastic stress solution associated to R^S and of the variables self-stresses σ_{ij}^{R*}) be contained within the yield surface.

However, it may be quite difficult in general to solve the abovementioned optimization problem. Thus it may be important to furnish simple bounds which can be easily evaluated and accepted by engineering practice. Then a simple first bounding inequality may be written in the form:

$$u_a^R(P) \leq \frac{1}{2R^C}\left\{\int_V A_{ijhk}\sigma_{ij}^{RC}\sigma_{hk}^{RC}\,\mathrm{d}V + \frac{1}{S-1}\int_V A_{ijhk}\bar{\sigma}_{ij}^{RS}\bar{\sigma}_{hk}^{RS}\,\mathrm{d}V\right\} \tag{50}$$

where R^C is the static collapse value of R^S, σ_{ij}^{RC} are the selfstresses associated with R^C and S, $\bar{\sigma}_{ij}^{RS}$ are the shakedown safety factor and, respectively, the associated selfstresses.

A second way to define a simple bound may be obtained by supposing the value R^S coincident with the elastic limit R^E. In such a case, being $\sigma_{ij}^{R*} \equiv 0$, we have:

$$u_a^R(P) \le \frac{1}{2(S-1)R^E} \int_V A_{ijhk} \bar{\sigma}_{ij}^{RS} \bar{\sigma}_{hk}^{RS} \, dV. \tag{51}$$

It is obvious that the smaller value between the values (50) and (51) constitutes, in this limited context, the best bound that we may obtain.

Finally, it should be useful to note that all the preceeding results may be used for one- and two-dimensional continua by considering the appropriate generalized stress and strain components. Moreover, the one-parameter loading conditions may be treated in the same way since they may be considered as particular cases in the shakedown theory.

Two simple one-dimensional continua will be discussed, as examples, in the following section.

6. EXAMPLES

The bounds which follow are intended only to illustrate the simplified bounding technique based on inequalities (50) and (51).

As a first example we will investigate the two-span beam of Fig. 1a. The continuous beam is of uniform section, full plastic moment M_P, flexural elastic rigidity EI, and the loads F_1 and F_2 vary independently between the identical limits:

$$\begin{aligned} 0 &\le F_1 \le F_0 \\ 0 &\le F_2 \le F_0. \end{aligned} \tag{52}$$

The largest value of F_0 for which the beam system shakes down is given by the value (see e.g.[11] at p. 135):

$$F_S = \frac{96 M_P}{19 l} = 5{\cdot}05 \frac{M_P}{l}. \tag{53}$$

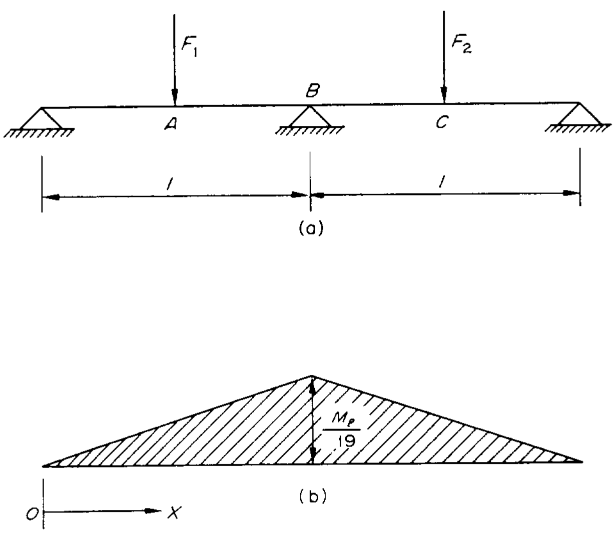

Fig. 1

The pattern of residual moments $\overline{M}_R^S$, when $F_0 = F_S$, is sketched in Fig. 1b and the value of the residual moment at B is given by the relation:

$$\overline{M}_R^S(B) = \frac{M_P}{19}. \tag{54}$$

A bound on the central residual displacement w_A^R of the point A, when $F_0 \leq F_S$, is desired. Thus the fictitious static system which we must chose is the one shown in Fig. 2a. Therefore, if we wish to make use of the inequality (50), F_1 must coincide with the static collapse load of the beam:

$$F_C = \frac{6M_P}{l}. \tag{55}$$

The pattern of residual moments M_R^C, when $F_1 = F_C$, is sketched in Fig. 2b and the value of the residual moment at B is given by the relation:

$$M_R^C(B) = \frac{7M_P}{16}. \tag{56}$$

Thus, the equivalent of the bound expression (50) in the present case may be written in the form:

$$w_A^R \leq \frac{l}{12M_P}\left\{2\int_0^l \left(\frac{7M_P}{16}\frac{x}{l}\right)^2 \frac{dx}{EI} + \frac{2}{S-1}\int_0^l \left(\frac{M_P}{19}\frac{x}{l}\right)^2 \frac{dx}{EI}\right\} \tag{57}$$

which gives the first bounding relationship:

$$w_A^R \leq \frac{M_P l^2}{18EI}\left\{\frac{49}{256} + \frac{1}{361(S-1)}\right\}. \tag{58}$$

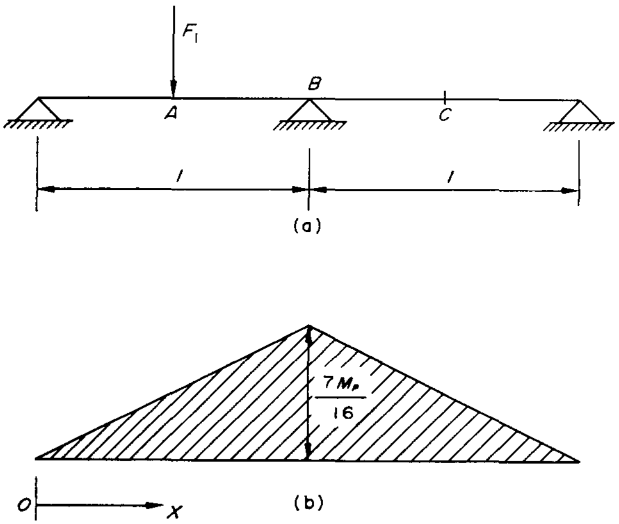

Fig. 2

A second bounding relationship may be developed by using inequality (51). In such a case, in the fictitious load condition shown in Fig. 2a, F_1 must be coincident with the elastic limit load:

$$F_E = \frac{64M_P}{13l} = 4{\cdot}92\frac{M_P}{l}\,. \tag{59}$$

Thus, the equivalent of the bound expression (51) may be written in the form:

$$w_A{}^R \leq \frac{13l}{128M_P}\left\{\frac{2}{S-1}\int_0^l\left(\frac{M_P}{19}\frac{x}{l}\right)^2\frac{\mathrm{d}x}{EI}\right\} \tag{60}$$

which gives the second bounding relationship:

$$w_A{}^R \leq \frac{13M_Pl^2}{192EI}\cdot\frac{1}{361(S-1)}\,. \tag{61}$$

By comparing relationships (58) and (61) is is easy to note that the first one gives the best bound for $1 \leq S \leq 1{\cdot}0032$. Thus, for $S \geq 1{\cdot}0032$ the best bound is given by the relationship (61).

Now, in order to compare these bounds to the actual response of the structure for a prefixed load cycle, we suppose that after increasing the load F_1 from zero to F_0, this load is fixed at the value F_0 while F_2 is allowed to fluctuate between zero and the same value F_0.

After developing a simple calculation, the residual displacement $w_A{}^R$ at point A may be written in the form:

$$\begin{aligned} w_A{}^R &= \frac{169F_0l^2}{1536EI} - \frac{13M_Pl^2}{24EI} \quad &\text{if} \quad \frac{64M_P}{13l} \leq F_0 \leq \frac{96M_P}{19l} \\ w_A{}^R &= 0 &\text{if} \quad 0 \leq F_0 \leq \frac{64M_P}{13l}\,. \end{aligned} \tag{62}$$

Since:

$$F_0 = \frac{F_S}{S} = \frac{96M_P}{19lS}\,. \tag{63}$$

Equations (62) may be rewritten in the form:

$$\begin{aligned} w_A{}^R &= \frac{13M_Pl^2}{912EI}\cdot\frac{39-38S}{S} \quad &\text{if} \quad 1 \leq S \leq \frac{39}{38} \\ w_A{}^R &= 0 &\text{if} \quad \frac{39}{38} \leq S \leq \infty \end{aligned} \tag{64}$$

The results of these calculations together with the bounding values (58) and (61) have been plotted on Fig. 3 as functions of the safety factor S.

The difference between the best bound value and the actual value of $w_A{}^R$ is quite large, but it may be reduced if an optimization technique of the general bound expression (48) is developed. In the limited context of the simplified inequalities (50) and (51) we can only expect to obtain the order of the residual displacement.

In order to show now that the results of this paper enable us also to bound the residual

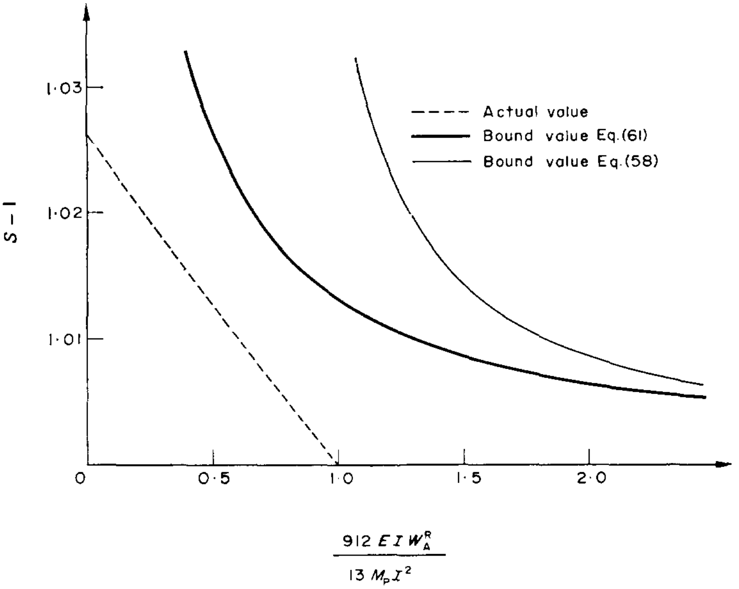

Fig. 3

displacements of structures subjected to one-parameter loading conditions; we will investigate, as a second example, the fixed ended beam of Fig. 4a. This beam is of uniform section, full plastic moment M_P, flexural elastic rigidity EI and the uniform load P varies between the limits:

$$0 \leq P \leq P_0 . \tag{65}$$

The largest value of P_0 for which the beam shakes-down coincides, in this case, with the static collapse value:

$$P_S \equiv P_C = \frac{16 M_P}{l^2} . \tag{66}$$

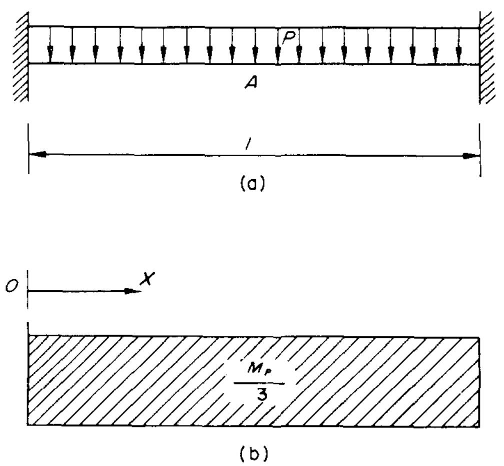

Fig. 4

The pattern of residual moments $\overline{M}_R{}^S$, when $P_0 = P_S$, is sketched in Fig. 4b and turns out to be of constant value:

$$\overline{M}_R{}^S(x) = \frac{M_P}{3}. \tag{67}$$

A bound on the central residual displacement $w_A{}^R$ of the point A, when $P_0 < P_S$, is desired.

Thus the fictitious static system which we must chose is the one shown in Fig. 5. Since in this case the residual moments $M_R{}^C$ which develop in the beam when F coincides with the static collapse value:

$$F_C = \frac{8M_P}{l} \tag{68}$$

vanish, the bound expressions (50) and (51) give the same result.

Thus the bounding relation may be written in the form:

$$w_A{}^R \leq \frac{l}{16M_P}\left\{\frac{1}{S-1}\int_0^l \left(\frac{M_P}{3}\right)^2 \frac{dx}{EI}\right\} \tag{69}$$

which gives:

$$w_A{}^R \leq \frac{M_P l^2}{144EI} \cdot \frac{1}{S-1}. \tag{70}$$

It is easy to show that the actual value of $w_A{}^R$ is given by the equations:

$$\begin{aligned} w_A{}^R &= \frac{P_0 l^4}{96EI} - \frac{M_P l^2}{8EI} \quad \text{if} \quad \frac{12M_P}{l^2} \leq P_0 \leq \frac{16M_P}{l^2} \\ w_A{}^R &= 0 \quad \text{if} \quad 0 \leq P_0 \leq \frac{12M_P}{l^2}. \end{aligned} \tag{71}$$

Thus, bearing in mind that:

$$P_0 = \frac{P_S}{S} = \frac{16M_P}{l^2 S} \tag{72}$$

equations (71) may be rewritten in the form:

$$\begin{aligned} w_A{}^R &= \frac{M_P l^2}{24EI} \cdot \frac{4-3S}{S} \quad \text{if} \quad 1 \leq S \leq \tfrac{4}{3} \\ w_A{}^R &= 0 \quad \text{if} \quad \tfrac{4}{3} \leq S \leq \infty. \end{aligned} \tag{73}$$

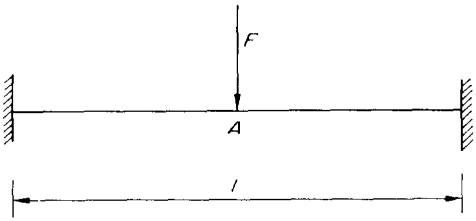

Fig. 5

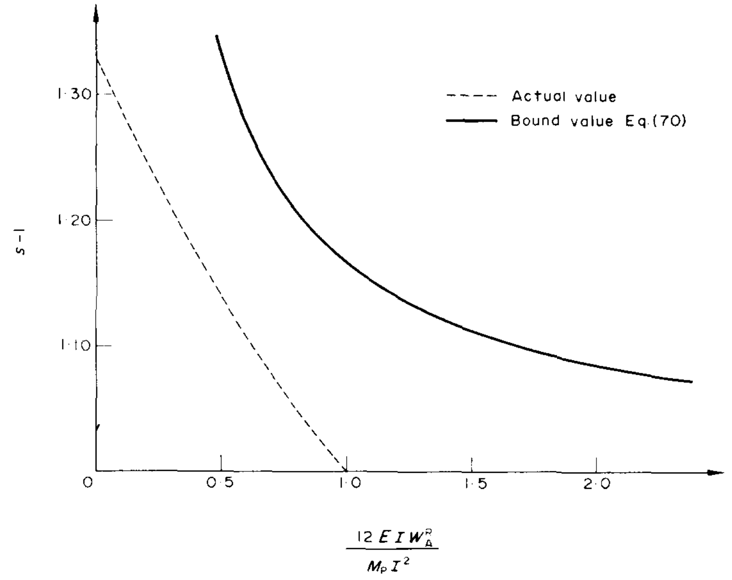

Fig. 6

The results of these calculations, together with the bounding value (70), have been plotted on Fig. 6 as functions of the safety factor S.

In this case also it may be noted that the difference between the bound curve and the actual curve is quite large.

7. CONCLUSIONS

In this paper we have discussed a method to obtain upper bounds on the residual displacement at a point of an elastic–plastic structure subject to variable loading.

The bound is obtained as the sum of displacements which derive from the elastic strain energy of two residual states of stress: the first one is associated with a fictitious loading condition whose distribution is entirely dictated by those properties of the actual displacement field for which a bound is desired, the second one is associated to the loading range in the prefixed interval of time.

The result applies on the usual assumptions of perfect plasticity and small strains.

On the same assumptions, a recent work of Ponter[7] provides an entirely different bounding principle which furnishes the bound on the global displacement at a point as the sum of an elastic deflection and of an additional displacement which derives from a residual state of stress.

This latter, as well as the elastic deflections, is associated with a loading programme which includes both the fictitious loading condition as well as the actual loading range.

Thus the result is entirely different and it is not easily comparable with the present derivation.

However it may be observed that, from a practical point of view, both the principles seem to provide similar answers in what concerns the bound approximation.

It may be also interesting to compare numerically these results with tlose that may be deduced by using the alternative formulations proposed by Vitiello[5] and Maier[6] in the field of the discrete models of piecewise linear elastoplastic structures. As in the latter work[6], it would be interesting to extend the present results to allow for hardening and geometric effect which may have a noteworthy importance in this area.

Acknowledgements—The present research was carried out with the financial support of C. N. R.

REFERENCES

1. E. Melan, Der Spannungszustand eines Hencky-Mises'schen Kontinuums bei Verändlicher Belastung. *Sitzungsberichte der Akademie Wissenschaften Wien*, Series 2 A, **147**, 73 (1938).
2. W. T. Koiter, General theorems for elastic–plastic solids, Vol. 1, Chapter 4. *Progress in Solid Mechanics*. Edited by I. N. Sneddon and R. Hill. North Holland (1960).
3. G. Maier, Shakedown theory in perfect elastoplasticity with associated and nonassociated flow-laws: a finite element linear programming approach. *Meccanica* **4** (3), 250 (1969).
4. G. Maier, A matrix structural theory of piecewise linear elastoplasticity with interacting yield planes. *Meccanica* **5** (1), 54 (1970).
5. E. Vitiello, Upper bounds to plastic strains in shakedown of structures subjected to cyclic loads. *Meccanica* **7** (3), 205 (1972).
6. G. Maier, A shakedown matrix theory allowing for workhardening and second order geometric effects. *International Symposium on Foundations of Plasticity* Warsaw (1972), Preprint edition. Noordhoff, p. 417.
7. A. R. S. Ponter, An upper bound on the small displacements of elastic, perfectly plastic structures. *J. appl. Mech.* **39**, 959 (1972).
8. D. C. Drucker, A more fundamental approach to plastic stress–strain relations. *Proc. First U.S. Nat. Congr. appl. Mech.* 487 (1951).
9. G. Maier, Complementary plastic work theorems in piecewise-linear elastoplasticity. *Int. J. Solids Struct.* **5**, 261 (1969).
10. A. Haar and Th. V. Karman, Zur Theorie der Spannungszustande in Plastischen und Sandartigen Medien. *Nach. der Wiss. zu Göttingen*, **18**, 204 (1909).
11. P. G. Hodge, A deformation bounding theorem for flow law plasticity. *Q. appl. Math.* **24**, 171 (1966).
12. A. R. S. Ponter and J. B. Martin, Some extremal properties and energy theorems for inelastic materials and their relationship to the deformation theory of plasticity. *J. Mech. Phys. Solids* **20**, 281 (1972).
13. J. Heyman, *Plastic Design of Frames*. Cambridge University Press (1971). *Applications*, Vol. 2.

Абстракт — Исследуются сплошные среды или структуры составленные из упругого идеально-пластического материала, подверженные действию циклических нагрузок, которые изменяются в пределах до разрушения. Дается теорема для ограничения остаточного прогиба.

Обсуждаются некоторые численные примеры.